

INVARIANT DIMENSIONS AND MAXIMALITY OF GEOMETRIC MONODROMY ACTION

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ABSTRACT. Let X be a smooth separated geometrically connected variety over \mathbb{F}_q and $f : Y \rightarrow X$ a smooth projective morphism. We compare the invariant dimensions of the ℓ -adic representation V_ℓ and the \mathbb{F}_ℓ -representation \bar{V}_ℓ of the geometric étale fundamental group of X arising from the sheaves $R^w f_* \mathbb{Q}_\ell$ and $R^w f_* \mathbb{Z}/\ell\mathbb{Z}$ respectively. These invariant dimension data is used to deduce a maximality result of the geometric monodromy action on V_ℓ whenever \bar{V}_ℓ is semisimple and ℓ is sufficiently large. We also provide examples for \bar{V}_ℓ to be semisimple for $\ell \gg 0$.

1. INTRODUCTION

Consider a smooth projective \mathbb{F}_q -morphism $f : Y \rightarrow X$, where X is a smooth separated geometrically connected \mathbb{F}_q -variety. Fix a geometric point $\bar{x}_0 : \text{Spec}(\bar{\mathbb{F}}_q) \rightarrow X$. For any prime $\ell \nmid q$ and integer w , $\mathcal{F}_\ell := R^w f_* \mathbb{Q}_\ell$ is a *lisse, pure of weight w* , \mathbb{Q}_ℓ -sheaf on X [De80] inducing an ℓ -adic representation of the *étale fundamental group* $\pi_1^{et}(X) := \pi_1^{et}(X, \bar{x}_0)$ on the stalk $\mathcal{F}_{\ell, \bar{x}_0} \cong H^w(Y_{\bar{x}_0}, \mathbb{Q}_\ell) =: V_\ell$,

$$(1) \quad \Phi_\ell : \pi_1^{et}(X) \rightarrow \text{GL}(V_\ell);$$

$\bar{\mathcal{F}}_\ell := R^w f_* \mathbb{Z}/\ell\mathbb{Z}$ is a locally constant sheaf on X inducing an \mathbb{F}_ℓ -representation on the stalk $\bar{\mathcal{F}}_{\ell, \bar{x}_0} \cong H^w(Y_{\bar{x}_0}, \mathbb{Z}/\ell\mathbb{Z}) =: \bar{V}_\ell$,

$$(2) \quad \phi_\ell : \pi_1^{et}(X) \rightarrow \text{GL}(\bar{V}_\ell).$$

The *geometric étale fundamental group* of X , $\pi_1^{et}(X_{\bar{\mathbb{F}}_q}) := \pi_1^{et}(X_{\bar{\mathbb{F}}_q}, \bar{x}_0)$, is a normal subgroup of $\pi_1^{et}(X)$ satisfying the exact sequence

$$(3) \quad 1 \rightarrow \pi_1^{et}(X_{\bar{\mathbb{F}}_q}) \rightarrow \pi_1^{et}(X) \rightarrow \text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \rightarrow 1$$

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so that any $x \in X(\mathbb{F}_q)$ induces a splitting i_x of (3). The *monodromy group* Γ_ℓ (resp. $\bar{\Gamma}_\ell$) and the *geometric monodromy group* Γ_ℓ^{geo} (resp. $\bar{\Gamma}_\ell^{\text{geo}}$) are defined to be the images of $\pi_1^{\text{et}}(X)$ and $\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})$ respectively in $\text{GL}(V_\ell)$ (resp. $\text{GL}(\bar{V}_\ell)$); their Zariski closures in GL_{V_ℓ} , denoted respectively by \mathbf{G}_ℓ and $\mathbf{G}_\ell^{\text{geo}}$, are called the *algebraic monodromy group* and the *algebraic geometric monodromy group* of Φ_ℓ .

Since $\text{Gal}(\bar{\mathbb{F}}_q/\mathbb{F}_q) \cong \hat{\mathbb{Z}}$ is abelian, the geometric monodromy groups Γ_ℓ^{geo} and $\mathbf{G}_\ell^{\text{geo}}$ are of particular interest by (3). Deligne has proved that the identity component of $\mathbf{G}_\ell^{\text{geo}}$ is a semisimple subgroup of GL_{V_ℓ} [De80, Cor. 1.3.9, Thm. 3.4.1(iii)]. Determining Γ_ℓ^{geo} (or $\bar{\Gamma}_\ell^{\text{geo}}$) and $\mathbf{G}_\ell^{\text{geo}}$ for families of curves (elliptic [Ha08]; hyperelliptic [La90s], [Yu96], [AP07]; trielliptic [AP07]) is of independent interest and also has applications to the arithmetic of function fields (see [Yu96], [Ac06]) and arithmetic geometry (see [Ch97], [Ko06a, Ko06b, Ko06c, Ko08]) over function fields. A crucial point is that for all sufficiently large ℓ , the geometric monodromy Γ_ℓ^{geo} is a *large* compact subgroup of $\mathbf{G}_\ell^{\text{geo}}(\mathbb{Q}_\ell)$. The motivation of this paper is to investigate the following large geometric monodromy conjecture. Let $\pi : \mathbf{G}_\ell^{\text{sc}} \rightarrow \mathbf{G}_\ell^{\text{geo}}$ be the natural morphism such that $\mathbf{G}_\ell^{\text{sc}}$ is the universal cover of the identity component of $\mathbf{G}_\ell^{\text{geo}}$.

Conjecture 1. *Let Φ_ℓ be the ℓ -adic representation defined in (1). Then $\pi^{-1}(\Gamma_\ell^{\text{geo}})$ is a hyperspecial maximal compact subgroup of $\mathbf{G}_\ell^{\text{sc}}(\mathbb{Q}_\ell)$ whenever ℓ is sufficiently large.*

Let us make a detour to the characteristic zero case. Suppose $f : Y \rightarrow X$ is not defined over \mathbb{F}_q , but over a subfield K of \mathbb{C} . Denote the w th Betti cohomology $H^w(Y_{\bar{x}_0}(\mathbb{C}), \mathbb{Q})$ by V , which is acted on by the topological fundamental group $\pi_1(X(\mathbb{C}))$. Since the geometric representation $\Phi_\ell : \pi_1^{\text{et}}(X_{\bar{K}}) \rightarrow \text{GL}(V_\ell)$ is arising from $\Phi : \pi_1(X(\mathbb{C})) \rightarrow \text{GL}(V)$ by the comparison theorem between Betti and étale cohomologies [SGA1, XII], [SGA4, XVI] and the identity component of algebraic monodromy group of Φ is semisimple over \mathbb{Q} [De71, Cor. 4.2.9], the geometric monodromy Γ_ℓ^{geo} is large in $\mathbf{G}_\ell^{\text{geo}}(\mathbb{Q}_\ell)$ for $\ell \gg 0$, thanks to [MVW84]. On the other hand, $\pi_1^{\text{et}}(X)$ satisfies (3) with \mathbb{F}_q replaced with K . Since $\text{Gal}(\bar{K}/K)$ is non-abelian, the monodromy groups $\Gamma_\ell \subset \mathbf{G}_\ell(\mathbb{Q}_\ell)$ are complicated and carry a lot of arithmetic information. If K is a number field and $X = \text{Spec}(K)$, then Φ_ℓ is a Galois representation of K arising from the smooth projective variety Y/K and the largeness of Γ_ℓ in $\mathbf{G}_\ell(\mathbb{Q}_\ell)$ for $\ell \gg 0$ follows from the remarkable conjectures of Hodge, Grothendieck, Tate, Mumford-Tate, and Serre [Se94, §11], see also [HL15a, §5]. The prototypical result in this direction is due to Serre [Se72], which states that for any non-CM elliptic curve Y , the monodromy Γ_ℓ on H^1 is $\text{GL}_2(\mathbb{Z}_\ell)$ for all sufficiently large

ℓ , see also [Ri76, Ri85], [Se85], [BGK03, BGK06, BGK10], [Ha11] for certain abelian varieties; [HL15b] for arbitrary abelian varieties; [Se98] for abelian representations; [HL14] for type A representations; and partial results [La95a], [Hu14] for arbitrary varieties. To get large Galois monodromy, one always needs handles on the invariants of V_ℓ and \bar{V}_ℓ . For example when Y is an abelian variety and $w = 1$, Faltings has proved that the Galois invariants of $V_\ell \otimes V_\ell^*$ and $\bar{V}_\ell \otimes \bar{V}_\ell^*$ depend essentially on the endomorphism ring $\text{End}(Y_{\bar{K}})$ if ℓ is sufficiently large [Fa83], [FW84] (the Tate conjecture). Since the Tate conjecture remains largely open, large Galois monodromy is presumably difficult.

Back to our setting $f : Y \rightarrow X$ over \mathbb{F}_q , the main idea of this paper is that there is a *cohomological* way, without resorting to the Tate conjecture, to compare the geometric invariant dimensions of $V_\ell^{\otimes m}$ and $\bar{V}_\ell^{\otimes m}$ for sufficiently large ℓ and sufficiently many m .

Theorem 2. *For any $m \in \mathbb{N}$, if ℓ is sufficiently large, then*

$$\dim_{\mathbb{F}_\ell}(\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\mathbb{Q}_\ell}(V_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}.$$

This is accomplished in §2 first assuming X is a curve by étale cohomology theory [SGA4, SGA4 $\frac{1}{2}$, SGA5], [Mi80], [FK87], [Fu11] and the remarkable theorems of Deligne [De74b, De80], Gabber [Ga83], and de Jong [dJ96], the general case then follows from that by space filling curves [Ka99] and ℓ -independence of \mathbf{G}_ℓ [Ch04].

Theorem 3. *If ϕ_ℓ is semisimple for all sufficiently large ℓ , then Conjecture 1 holds.*

Theorem 3 is proved in §3 by a recent result of Cadoret and Tamagawa on $\bar{\Gamma}_\ell^{\text{geo}}$ [CT15], the group theoretic techniques we employed and developed in [HL14], and exploiting the invariant dimension data (Theorem 2 and Corollary 2.3). The \mathbb{F}_ℓ -semisimplicity hypothesis of Theorem 3 holds if X is a curve and the fibers of f are curves or abelian varieties [Za74a, Za74b]. It is suggestive that the hypothesis holds in general because the invariant dimensions of Γ_ℓ^{geo} and $\bar{\Gamma}_\ell^{\text{geo}}$ are alike (Theorem 2) and Γ_ℓ^{geo} is semisimple on V_ℓ . Nevertheless, we provide in §4 some examples for the hypothesis to hold.

2. INVARIANT DIMENSIONS

The notation in §1 remains in force. Embed $\bar{\mathbb{Z}}[\frac{1}{q}]$ into $\bar{\mathbb{Z}}_\ell$ with unique maximal ideal \mathfrak{m}_ℓ . The common dimension of V_ℓ for all ℓ (not dividing q) is also equal to the common dimension of \bar{V}_ℓ for all sufficiently large ℓ [Ga83]. Then whenever $\dim_{\mathbb{F}_\ell} \bar{V}_\ell = \dim_{\mathbb{Q}_\ell} V_\ell$, one obtains

$$(4) \quad \dim_{\mathbb{F}_\ell}(\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} \geq \dim_{\mathbb{Q}_\ell}(V_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}$$

for any $m \in \mathbb{N}$ by identifying Γ_ℓ^{geo} as a subgroup of $\text{GL}(H^i(Y_{\bar{x}_0}, \mathbb{Z}_\ell))$, the reduction map $\text{GL}(H^i(Y_{\bar{x}_0}, \mathbb{Z}_\ell)) \rightarrow \text{GL}(\bar{V}_\ell)$, and Lemma 2.1.

Lemma 2.1. *Let F be a characteristic 0 non-Archimedean local field with \mathcal{O}_F the ring of integers. Let M be a free \mathcal{O}_F -module of finite rank. If W is an F -subspace of $M \otimes_{\mathcal{O}_F} F$, then $W \cap M$ is a direct summand of M .*

Proof. Since \mathcal{O}_F is a PID and \mathcal{O}_F/I is finite for any non-zero ideal I , the finitely generated module $M/W \cap M$ is a direct sum of a free submodule and a torsion submodule. If $x \in M$ maps to a torsion element in $M/W \cap M$, then $k \cdot x \in W \cap M$ for some $k \in \mathbb{N}$. This implies $x \in W$ because F is of characteristic 0. Hence, $M/W \cap M$ is free and $W \cap M$ is a direct summand of M . \square

Let d be the dimension of X . Since X is smooth, \mathcal{F}_ℓ is lisse, and $\tilde{\mathcal{F}}_\ell$ is locally constant, we obtain perfect pairings by Poincaré duality [Mi80, VI Thm. 11.1] which is compatible with the action of *geometric Frobenius* Fr_q :

$$(5) \quad \begin{aligned} H^i(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell) \times H_c^{2d-i}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell^\vee) &\rightarrow \mathbb{Q}_\ell(-d); \\ H^i(X_{\bar{\mathbb{F}}_q}, \tilde{\mathcal{F}}_\ell) \times H_c^{2d-i}(X_{\bar{\mathbb{F}}_q}, \tilde{\mathcal{F}}_\ell^\vee) &\rightarrow \mathbb{F}_\ell(-d). \end{aligned}$$

The geometric invariants admit the following descriptions:

$$(6) \quad \begin{aligned} V_\ell^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})} &= H^0(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell); \\ \bar{V}_\ell^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})} &= H^0(X_{\bar{\mathbb{F}}_q}, \tilde{\mathcal{F}}_\ell). \end{aligned}$$

Without loss of generality, assume x_0 is an \mathbb{F}_q -rational point of X that induces a *splitting* of (3). Then the *multiset* A' of the Fr_q -eigenvalues on $V_\ell := H^w(Y_{\bar{x}_0}, \mathbb{Q}_\ell)$ are independent of ℓ [De74b] and pure of weight w [De80]. It follows that the eigenvalues on $H^0(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell)$ belong to those on V_ℓ by the splitting and (6). One also sees by the same token that the eigenvalues on $H^0(X_{\bar{\mathbb{F}}_q}, \tilde{\mathcal{F}}_\ell)$ belong to the reduction modulo \mathfrak{m}_ℓ of the eigenvalues on V_ℓ whenever $\dim_{\mathbb{F}_\ell} \bar{V}_\ell = \dim_{\mathbb{Q}_\ell} V_\ell$. Define A to be the following multiset:

$$(7) \quad A := \{q^d \alpha^{-1} : \alpha \in A'\}.$$

We conclude by (5) and above that the numbers in A are pure of weight $2d - w$, the eigenvalues of $H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \mathcal{F}_\ell^\vee)$ is a sub-multiset of A , and the eigenvalues of $H_c^{2d}(X_{\bar{\mathbb{F}}_q}, \tilde{\mathcal{F}}_\ell^\vee)$ is a sub-multiset of the reduction modulo \mathfrak{m}_ℓ of A for $\ell \gg 0$.

Theorem 2. *For any $m \in \mathbb{N}$, if ℓ is sufficiently large, then*

$$\dim_{\mathbb{F}_\ell} (\bar{V}_\ell^{\otimes m})^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})} = \dim_{\mathbb{Q}_\ell} (V_\ell^{\otimes m})^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})}.$$

Proof. Step I. Assume X is a (geometrically connected) curve, i.e., $d = 1$. If U is an affine open subscheme of X containing x_0 , then $\pi_1^{et}(U)$ surjects onto $\pi_1^{et}(X)$ [Fu11, Prop. 3.3.4(i)] and we obtain a commutative diagram:

$$(8) \quad \begin{array}{ccc} \pi_1^{et}(U) & \longrightarrow & \mathrm{GL}(V_\ell) \\ \downarrow & & \downarrow = \\ \pi_1^{et}(X) & \longrightarrow & \mathrm{GL}(V_\ell) \end{array}$$

Hence, we may further assume X is an affine curve.

Step II. Let $e > 0$ be the relative dimension of $f : Y \rightarrow X$. Then the dimension of Y is $e + 1$. Let Y^c be a compactification of $Y_{\bar{\mathbb{F}}_q}$. Then Y^c admits a *simplicial scheme* Y , projective and smooth over $\bar{\mathbb{F}}_q$ and an augmentation $Y \rightarrow Y^c$ which is a *proper hypercovering* of Y^c (see [dJ96, §1]). This induces a spectral sequence

$$(9) \quad E_1^{i,j} := H^j(Y_i, \mathbb{Z}/\ell\mathbb{Z}) \Rightarrow H^{i+j}(Y^c, \mathbb{Z}/\ell\mathbb{Z})$$

by [Co03, (6.3), Thm. 7.9] (see also [De74a]). Let B' be the multiset consisting of all the Fr_q -eigenvalues on $H^j(Y_i, \mathbb{Q}_\ell)$ for all $(i, j) \in \mathbb{Z}_{\geq 0}^2$ satisfying $i + j = 2(e + 1) - (1 + w)$. Since Y_i is smooth projective for all i , the multiset B' is mixed of weight $\leq 2(e + 1) - (1 + w)$ and is independent of ℓ [De74b, De80]. Since there are only finitely many such (i, j) , the Fr_q -eigenvalues on

$$H^{2(e+1)-(1+w)}(Y^c, \mathbb{Z}/\ell\mathbb{Z}) =: H_c^{2(e+1)-(1+w)}(Y_{\bar{\mathbb{F}}_q}, \mathbb{Z}/\ell\mathbb{Z})$$

belong to the reduction (modulo \mathfrak{m}_ℓ) of B' for $\ell \gg 0$ by [Ga83] and the biregular spectral sequence (9). Since Y is smooth, the reduction (modulo \mathfrak{m}_ℓ) of the multiset (mixed of weight $\geq 1 + w$)

$$B'' := \{q^{e+1}\beta^{-1} : \beta \in B'\}$$

contains all the Fr_q -eigenvalues on $H^{1+w}(Y_{\bar{\mathbb{F}}_q}, \mathbb{Z}/\ell\mathbb{Z})$ for $\ell \gg 0$ by Poincare duality. Since the spectral sequence

$$E_2^{i,j} := H^i(X_{\bar{\mathbb{F}}_q}, R^j f_* \mathbb{Z}/\ell\mathbb{Z}) \Rightarrow H^{i+j}(Y_{\bar{\mathbb{F}}_q}, \mathbb{Z}/\ell\mathbb{Z})$$

degenerates on page 2 (as X is an affine curve), $E_2^{1,w} = H^1(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell)$ is a sub-quotient of $H^{1+w}(Y_{\bar{\mathbb{F}}_q}, \mathbb{Z}/\ell\mathbb{Z})$. Thus, the eigenvalues on $H^1(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell)$ belong to the reduction of B'' for $\ell \gg 0$. Then we conclude that the multiset (mixed of weight $\leq 1 - w$)

$$(10) \quad B := \{q\beta^{-1} : \beta \in B''\}$$

after reduction contains all the eigenvalues on $H_c^1(X_{\bar{\mathbb{F}}_q}, \bar{\mathcal{F}}_\ell^\vee)$ for $\ell \gg 0$ by X smooth and Poincare duality again.

Step III. By the Lefschetz trace formula on the lisse sheaf \mathcal{F}_ℓ^\vee and the locally constant sheaf $\tilde{\mathcal{F}}_\ell^\vee$ on $X_{\mathbb{F}_q}$ [Mi80, VI Thm. 13.4], we obtain

$$(11) \quad \sum_{x \in X(\mathbb{F}_{q^k})} \text{Tr}(\text{Fr}_q^k : H^w(Y_{\bar{x}}, \mathbb{Q}_\ell)^\vee) = \sum_{i=0}^2 (-1)^i \text{Tr}(\text{Fr}_q^k : H_c^i(X_{\mathbb{F}_q}, \mathcal{F}_\ell^\vee));$$

$$\sum_{x \in X(\mathbb{F}_{q^k})} \text{Tr}(\text{Fr}_q^k : H^w(Y_{\bar{x}}, \mathbb{Z}/\ell\mathbb{Z})^\vee) = \sum_{i=0}^2 (-1)^i \text{Tr}(\text{Fr}_q^k : H_c^i(X_{\mathbb{F}_q}, \tilde{\mathcal{F}}_\ell^\vee))$$

for all $k \in \mathbb{N}$. Since $Y_{\bar{x}}$ is smooth projective, the Frobenius action on $H^w(Y_{\bar{x}}, \mathbb{Z}/\ell\mathbb{Z})^\vee$ factors through $H^w(Y_{\bar{x}}, \mathbb{Q}_\ell)^\vee$ for $\ell \gg 0$ [Ga83]. Hence, the reduction of the first local sum is equal to the second local sum for $\ell \gg 0$. Since $H_c^0 = 0$ by X affine, we obtain

$$(12) \quad \sum_{i=1}^2 (-1)^i \overline{\text{Tr}(\text{Fr}_q^k : H_c^i(X_{\mathbb{F}_q}, \mathcal{F}_\ell^\vee))} = \sum_{i=1}^2 (-1)^i \text{Tr}(\text{Fr}_q^k : H_c^i(X_{\mathbb{F}_q}, \tilde{\mathcal{F}}_\ell^\vee))$$

for $\ell \gg 0$ by reduction. Denote the reductions of A (7) and B (10) by \bar{A} and \bar{B} respectively and the following:

- $\{\alpha_1, \dots, \alpha_r\} \subset \bar{A}$ the multiset of the Fr_q -eigenvalues on $H_c^2(X_{\mathbb{F}_q}, \tilde{\mathcal{F}}_\ell^\vee)$;
- $\{\beta_1, \dots, \beta_s\} \subset \bar{B}$ the multiset of the Fr_q -eigenvalues on $H_c^1(X_{\mathbb{F}_q}, \tilde{\mathcal{F}}_\ell^\vee)$;
- $\{a_1, \dots, a_t\} \subset \bar{A}$ the multiset of reduction of the Fr_q -eigenvalues on $H_c^2(X_{\mathbb{F}_q}, \mathcal{F}_\ell^\vee)$;
- $\{b_1, \dots, b_u\}$ the multiset of reduction of the Fr_q -eigenvalues on $H_c^1(X_{\mathbb{F}_q}, \mathcal{F}_\ell^\vee)$.

Note that $r, t \leq |A|$, $s \leq |B|$, and the number u is independent of ℓ [Ka83]. It follows from (12) that the above eigenvalues (in \mathbb{F}_ℓ^*) satisfy

$$(13) \quad a_1^k + \dots + a_t^k + \beta_1^k + \dots + \beta_s^k = \alpha_1^k + \dots + \alpha_r^k + b_1^k + \dots + b_u^k$$

for all $k \in \mathbb{N}$. If $\ell > |A| + \max\{|B|, u\}$, then by Lemma 2.2 the two multisets coincide:

$$(14) \quad \{a_1, \dots, a_t, \beta_1, \dots, \beta_s\} = \{\alpha_1, \dots, \alpha_r, b_1, \dots, b_u\}.$$

Since A is pure of weight $2 - w$ and B is mixed of weight $\leq 1 - w$ (Step II), $\bar{A} \cap \bar{B} = \emptyset$ for $\ell \gg 0$ which implies $t \geq r$ by (14). Together with (4), (5), and (6), we obtain

$$(15) \quad \dim_{\mathbb{F}_\ell} \bar{V}_\ell^{\pi_1^{et}(X_{\mathbb{F}_q})} = \dim_{\mathbb{Q}_\ell} V_\ell^{\pi_1^{et}(X_{\mathbb{F}_q})}$$

for all sufficiently large ℓ .

Step IV. Since $f : Y \rightarrow X$ is smooth projective, the natural morphism

$$Y^{[m]} := \overbrace{Y \times_X Y \times_X \cdots \times_X Y}^{m \text{ terms}} \rightarrow X$$

is still smooth projective with the fiber

$$(16) \quad (Y^{[m]})_{\bar{x}_0} = \prod_{\bar{x}_0}^m Y_{\bar{x}_0}$$

inducing the representations $W_\ell := H^{mw}((Y^{[m]})_{\bar{x}_0}, \mathbb{Q}_\ell)$ and $\bar{W}_\ell := H^{mw}((Y^{[m]})_{\bar{x}_0}, \mathbb{Z}/\ell\mathbb{Z})$ of $\pi_1^{et}(X)$. For all sufficiently large ℓ , we have

$$(17) \quad \dim_{\mathbb{F}_\ell} \bar{W}_\ell^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\mathbb{Q}_\ell} W_\ell^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}$$

by (15). Since the representation $V_\ell^{\otimes m}$ (resp. $\bar{V}_\ell^{\otimes m}$) is a direct summand of the representation W_ℓ (resp. \bar{W}_ℓ) by (16) and the Künneth isomorphism, we obtain by (17) that

$$(18) \quad \dim_{\mathbb{F}_\ell} (\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\mathbb{Q}_\ell} (V_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})}$$

holds for all sufficiently large ℓ . This proves Theorem 2 when X is a curve.

Step V. For general smooth geometrically connected X , it suffices to prove Theorem 2 for quasi-projective X (see (8)). If $C \subset X$ (containing x_0) is a smooth geometrically connected curve over \mathbb{F}_q , then

$$\Psi_\ell : \pi_1^{et}(C) \rightarrow \mathrm{GL}(V_\ell)$$

factors through Φ_ℓ for all ℓ . Denote by $\Lambda_\ell^{\mathrm{geo}}$ and $\mathbf{H}_\ell^{\mathrm{geo}}$ respectively the geometric monodromy group and the algebraic geometric monodromy group of Ψ_ℓ . Choose ℓ_0 such that the dimension of $\mathbf{G}_{\ell_0}^{\mathrm{geo}}$ is the largest. By [Ka99, Cor. 7, Thm. 8], there exists a space filling curve $C \subset X$ (smooth, geometrically connected, containing x_0 , over \mathbb{F}_q) satisfying

$$(19) \quad \mathbf{H}_{\ell_0}^{\mathrm{geo}} = \mathbf{G}_{\ell_0}^{\mathrm{geo}}.$$

Since the system $\{\Psi_\ell\}$ is pure of weight w and is semisimple on the geometric étale fundamental group $\pi_1^{et}(C_{\bar{\mathbb{F}}_q})$ (Deligne), the identity component $(\mathbf{H}_\ell^{\mathrm{geo}})^\circ$ (semisimple) is isomorphic to the derived group of the identity component of the algebraic monodromy group of the semisimplification of Ψ_ℓ for all ℓ by (3). This implies $(\mathbf{H}_\ell^{\mathrm{geo}})^\circ \times \mathbb{C}$ is independent of ℓ by applying [Ch04, Thm. 1.4] to the semisimplification of the system $\{\Psi_\ell\}$. In particular, the dimension of $\mathbf{H}_\ell^{\mathrm{geo}}$ is independent of ℓ . Since we have

$$\dim \mathbf{H}_\ell^{\mathrm{geo}} = \dim \mathbf{H}_{\ell_0}^{\mathrm{geo}} = \dim \mathbf{G}_{\ell_0}^{\mathrm{geo}} \geq \dim \mathbf{G}_\ell^{\mathrm{geo}}$$

and the groups $\mathbf{H}_\ell^{\mathrm{geo}} \subset \mathbf{G}_\ell^{\mathrm{geo}}$ (by Ψ_ℓ factors through Φ_ℓ) have the same number of connected components (by (19) and [LP95, Prop. 2.2(iii)])

for all ℓ , we obtain $\mathbf{H}_\ell^{\text{geo}} = \mathbf{G}_\ell^{\text{geo}}$ for all ℓ . Since (a) $\Lambda_\ell^{\text{geo}}$ (resp. Γ_ℓ^{geo}) is Zariski dense in $\mathbf{H}_\ell^{\text{geo}}$ (resp. $\mathbf{G}_\ell^{\text{geo}}$) and (b) Ψ_ℓ factors through Φ_ℓ , we obtain

$$\begin{aligned} \dim_{\mathbb{F}_\ell}(\bar{V}_\ell^{\otimes m})^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})} &\stackrel{(b)}{\leq} \dim_{\mathbb{F}_\ell}(\bar{V}_\ell^{\otimes m})^{\pi_1^{\text{et}}(C_{\bar{\mathbb{F}}_q})} \stackrel{(18)}{=} \dim_{\mathbb{Q}_\ell}(V_\ell^{\otimes m})^{\pi_1^{\text{et}}(C_{\bar{\mathbb{F}}_q})} \\ &\stackrel{(a)}{=} \dim_{\mathbb{Q}_\ell}(V_\ell^{\otimes m})^{\mathbf{H}_\ell^{\text{geo}}(\mathbb{Q}_\ell)} = \dim_{\mathbb{Q}_\ell}(V_\ell^{\otimes m})^{\mathbf{G}_\ell^{\text{geo}}(\mathbb{Q}_\ell)} \stackrel{(a)}{=} \dim_{\mathbb{Q}_\ell}(V_\ell^{\otimes m})^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})} \end{aligned}$$

for $\ell \gg 0$. We are done by (4). \square

Lemma 2.2. *Suppose $a_1, \dots, a_m, b_1, \dots, b_n \in \bar{\mathbb{F}}_\ell^*$ satisfying $\max\{m, n\} < \ell$ and*

$$(20) \quad a_1^k + \dots + a_m^k = b_1^k + \dots + b_n^k$$

for all $1 \leq k \leq \max\{m, n\}$. Then the two multisets $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_n\}$ coincide.

Proof. First assume $m = n$. Let x_1, \dots, x_m be indeterminate variables. Denote the elementary symmetric polynomials in x_1, \dots, x_m by e_1, \dots, e_m and $x_1^k + \dots + x_m^k$ by p_k . The *Newton's identities* imply

$$e_1, \dots, e_m \in \mathbb{Z}\left[\frac{1}{m!}\right](p_1, \dots, p_m).$$

Hence, $e_k(a_1, \dots, a_m) = e_k(b_1, \dots, b_m)$ for all $1 \leq k \leq m$ by (20) and $m < \ell$. We conclude that $\{a_1, \dots, a_m\} = \{b_1, \dots, b_m\}$ by constructing a degree m polynomial in $\bar{\mathbb{F}}_\ell[t]$ whose roots are exactly a_1, \dots, a_m (resp. b_1, \dots, b_m).

Suppose $m > n$. Let b_{n+1}, \dots, b_m be all zeros. Then some a_i is zero by the above case, which contradicts $a_i \in \bar{\mathbb{F}}_\ell^*$. \square

Corollary 2.3. *For any $m \in \mathbb{N}$, if ℓ is sufficiently large, then*

$$\begin{aligned} \dim_{\mathbb{F}_\ell}(\text{Sym}^m \bar{V}_\ell)^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})} &= \dim_{\mathbb{Q}_\ell}(\text{Sym}^m V_\ell)^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})}; \\ \dim_{\mathbb{F}_\ell}(\text{Alt}^m \bar{V}_\ell)^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})} &= \dim_{\mathbb{Q}_\ell}(\text{Alt}^m V_\ell)^{\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})}. \end{aligned}$$

Proof. Since the left hand side is always greater than or equal to the right hand side of the equation and the representations $\bar{V}_\ell^{\otimes m}$ and $V_\ell^{\otimes m}$ contain respectively $\text{Sym}^m \bar{V}_\ell$ and $\text{Sym}^m V_\ell$ (resp. $\text{Alt}^m \bar{V}_\ell$ and $\text{Alt}^m V_\ell$) as direct summands, the corollary follows from Theorem 2. \square

3. MAXIMALITY

If $X'_{\bar{\mathbb{F}}_q}$ is a connected finite étale cover of $X_{\bar{\mathbb{F}}_q}$, then $\pi_1^{\text{et}}(X'_{\bar{\mathbb{F}}_q})$ is a finite index subgroup of $\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})$. Since $X'_{\bar{\mathbb{F}}_q} \rightarrow X_{\bar{\mathbb{F}}_q}$ is always defined over some finite extension $\mathbb{F}_{q'}$ of \mathbb{F}_q (e.g., $X'_{\bar{\mathbb{F}}_q} \rightarrow X_{\mathbb{F}_{q'}}$) which does not affect the geometric monodromy and the restriction of a semisimple

representation to a normal subgroup is still semisimple, it suffices to prove Theorem 3 by considering the base change

$$Y \times_X X'_{\mathbb{F}_{q'}} \rightarrow X'_{\mathbb{F}_{q'}}$$

of $f : Y \rightarrow X$ by a connected finite Galois étale cover $X'_{\mathbb{F}_{q'}} \rightarrow X_{\mathbb{F}_{q'}} \rightarrow X$. Hence, we assume from now on that the algebraic geometric monodromy group $\mathbf{G}_\ell^{\text{geo}}$ is *connected* for all ℓ by taking a connected finite étale cover of X [LP95, Prop. 2.2(ii)]. Let n be the common dimension of V_ℓ for all ℓ , which is also the common dimension of \bar{V}_ℓ for $\ell \gg 0$.

Theorem 3. *If ϕ_ℓ is semisimple for all sufficiently large ℓ , then Conjecture 1 holds.*

Proof. Step I. For any subgroup $\bar{\Gamma}$ of $\text{GL}_n(\mathbb{F}_\ell)$, denote by $\bar{\Gamma}^+$ the (normal) subgroup of $\bar{\Gamma}$ that is generated by $\bar{\Gamma}[\ell]$, the subset of order ℓ elements of $\bar{\Gamma}$. By taking some connected finite Galois étale cover of X , we may assume $\bar{\Gamma}_\ell^{\text{geo}} = (\bar{\Gamma}_\ell^{\text{geo}})^+$ [CT15, Prop. 3.2, Thm. 1.1], $\bar{\Gamma}_\ell^{\text{geo}}$ is semisimple on \bar{V}_ℓ , and $\mathbf{G}_\ell^{\text{geo}}$ is connected for all sufficiently large ℓ . Since $n = \dim_{\mathbb{F}_\ell} \bar{V}_\ell$ for $\ell \gg 0$, there exists an *exponentially generated* subgroup $\bar{\mathbf{S}}_\ell$ of $\text{GL}_{\bar{V}_\ell}$ such that $\bar{\Gamma}_\ell^{\text{geo}} = \bar{\mathbf{S}}_\ell(\mathbb{F}_\ell)^+$ for all $\ell \gg 0$ by Nori [No87, Thm. B]. The Nori subgroup $\bar{\mathbf{S}}_\ell$ is connected and an extension of semisimple by unipotent. Since $\bar{\Gamma}_\ell^{\text{geo}}$ is semisimple on \bar{V}_ℓ for $\ell \gg 0$, $\bar{\mathbf{S}}_\ell$ is connected semisimple for $\ell \gg 0$. Let $\bar{\mathbf{S}}_\ell^{\text{sc}} \rightarrow \bar{\mathbf{S}}_\ell$ be the universal covering of $\bar{\mathbf{S}}_\ell$. The representation

$$\bar{\mathbf{S}}_\ell^{\text{sc}} \times \bar{\mathbb{F}}_\ell \rightarrow \bar{\mathbf{S}}_\ell \times \bar{\mathbb{F}}_\ell \hookrightarrow \text{GL}_{\bar{V}_\ell \times \bar{\mathbb{F}}_\ell}$$

can be lifted to a representation of some simply-connected *Chevalley scheme* over \mathbb{Z} for all $\ell \gg 0$ [Se86] (see [EHK12, Thm. 27]),

$$(21) \quad \rho_{\ell, \mathbb{Z}} : \mathbf{H}_{\ell, \mathbb{Z}} \rightarrow \text{GL}_{V_{\mathbb{Z}}}.$$

Step II. We would like to study the invariants of $\bar{\mathbf{S}}_\ell$ on $\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell$. Let us recall the construction of $\bar{\mathbf{S}}_\ell$. Define $\exp(x)$ and $\log(x)$ by

$$(22) \quad \exp(x) = \sum_{i=0}^{\ell-1} \frac{x^i}{i!} \quad \text{and} \quad \log(x) = - \sum_{i=1}^{\ell-1} \frac{(1-x)^i}{i}.$$

For all sufficiently large ℓ , $\bar{\mathbf{S}}_\ell$ is the Zariski closure in $\text{GL}_{\bar{V}_\ell} \cong \text{GL}_{n, \mathbb{F}_\ell}$ of the subgroup generated by the one-parameter subgroup

$$(23) \quad t \mapsto x^t := \exp(t \cdot \log(x))$$

for all $x \in \bar{\Gamma}_\ell^{\text{geo}}[\ell]$ (the order ℓ elements) [No87]. When $\ell > n$, x is unipotent and $\log(x)$ is nilpotent by (22). Identify $\bar{V}_\ell \otimes \bar{\mathbb{F}}_\ell$ with $\bar{\mathbb{F}}_\ell^n$, then every entry of the matrix $x^t \in \text{GL}_n(\bar{\mathbb{F}}_\ell[t])$ is a polynomial of degree less than n^2 by (22) and (23). Similarly, the action of x^t

on $\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell$ can be identified with an element of $\mathrm{GL}_{n^m}(\bar{\mathbb{F}}_\ell[t])$ whose entries are polynomials of degree less than n^2m . Consider an invariant $v \in (\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = (\bar{V}_\ell^{\otimes m})^{\bar{\Gamma}_\ell^{\mathrm{geo}}}$, then the equation in $\bar{\mathbb{F}}_\ell[t]^{n^m}$ below

$$x^t \cdot v = v$$

has at least ℓ distinct roots $t = 0, 1, \dots, \ell - 1$ because $id, x, \dots, x^{\ell-1} \in \bar{\Gamma}_\ell^{\mathrm{geo}}$. This implies $x^t \cdot v \equiv v$ when $\ell \geq n^2m$. Hence, we obtain $v \in (\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\bar{\mathbf{S}}_\ell}$ when $\ell \geq n^2m$ by the construction of $\bar{\mathbf{S}}_\ell$. Since $\bar{\Gamma}_\ell^{\mathrm{geo}} = (\bar{\Gamma}_\ell^{\mathrm{geo}})^+$ [CT15] is a subgroup of $\bar{\mathbf{S}}_\ell$ for $\ell \gg 0$, we obtain

$$\dim_{\mathbb{F}_\ell}(\bar{V}_\ell^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\bar{\mathbb{F}}_\ell}(\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\bar{\mathbf{S}}_\ell}$$

for $\ell \gg 0$. It follows that

$$\dim_{\mathbb{F}_\ell}((\oplus^n \bar{V}_\ell)^{\otimes m})^{\pi_1^{et}(X_{\bar{\mathbb{F}}_q})} = \dim_{\bar{\mathbb{F}}_\ell}((\oplus^n \bar{V}_\ell)^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\bar{\mathbf{S}}_\ell}$$

for $\ell \gg 0$. By Corollary 2.3 and embedding \mathbb{Q}_ℓ into \mathbb{C} , we conclude for $\ell \gg 0$ that

$$(24) \quad \dim_{\mathbb{C}}(\mathrm{Sym}^m(\oplus^n V_\ell) \otimes \mathbb{C})^{\mathbf{G}_\ell^{\mathrm{geo}}} = \dim_{\bar{\mathbb{F}}_\ell}(\mathrm{Sym}^m(\oplus^n \bar{V}_\ell) \otimes \bar{\mathbb{F}}_\ell)^{\bar{\mathbf{S}}_\ell}.$$

Step III. Denote the base change of (21) to \mathbb{C} by $\rho_{\ell, \mathbb{C}} : \mathbf{H}_{\ell, \mathbb{C}} \rightarrow \mathrm{GL}_{V_{\mathbb{C}}}$. For fixed $m \in \mathbb{N}$, we obtain by Step I and (24) that for $\ell \gg 0$,

$$(25) \quad \dim_{\mathbb{C}}(\mathrm{Sym}^m(\oplus^n V_\ell) \otimes \mathbb{C})^{\mathbf{G}_\ell^{\mathrm{geo}}} = \dim_{\mathbb{C}}(\mathrm{Sym}^m(\oplus^n V_{\mathbb{C}}))^{\mathbf{H}_{\ell, \mathbb{C}}}.$$

Since there are finitely many connected semisimple subgroup of $\mathrm{GL}_{n, \mathbb{C}}$ (up to isomorphism), (25) holds for all $m \in \mathbb{N}$ when ℓ is sufficiently large. Identify $V_\ell \otimes \mathbb{C}$ with $V_{\mathbb{C}}$. Then the (Noetherian) graded rings

$$R = \mathbb{C}[\oplus^n V_{\mathbb{C}}]^{\mathbf{G}_\ell^{\mathrm{geo}}} \quad \text{and} \quad R' = \mathbb{C}[\oplus^n V_{\mathbb{C}}]^{\mathbf{H}_{\ell, \mathbb{C}}}$$

have the same *Hilbert polynomial*, hence the same *Krull dimension* for $\ell \gg 0$. Since $\dim_{\mathrm{Krull}} R = n^2 - \dim \mathbf{G}_\ell^{\mathrm{geo}}$ and $\dim_{\mathrm{Krull}} R' = n^2 - \dim \rho_{\ell, \mathbb{C}}(\mathbf{H}_{\ell, \mathbb{C}})$ (for example [LP90, §0]), we conclude by the lifting (21) that for all $\ell \gg 0$,

$$(26) \quad \dim \mathbf{G}_\ell^{\mathrm{geo}} = \dim \bar{\mathbf{S}}_\ell.$$

Step IV. Suppose $\ell \geq 5$. For any compact subgroup $\Gamma \subset \mathrm{GL}_n(\mathbb{Q}_\ell)$ (resp. $\bar{\Gamma} \subset \mathrm{GL}_n(\bar{\mathbb{F}}_\ell)$), we defined the ℓ -dimension $\dim_\ell \Gamma$ (resp. $\dim_\ell \bar{\Gamma}$) in [HL14, §2] satisfying the following properties:

- (i) \dim_ℓ is additive on short exact sequences;
- (ii) \dim_ℓ vanishes for pro-solvable groups and finite simple groups that are not of Lie type in characteristic ℓ ;
- (iii) if $\bar{\Gamma}$ is a finite simple group of Lie type in characteristic ℓ , then there exists some connected adjoint semisimple group $\bar{\mathbf{S}}/\mathbb{F}_\ell$ such that $\bar{\Gamma}$ is isomorphic to the derived group of $\bar{\mathbf{S}}(\mathbb{F}_\ell)$ and we define $\dim_\ell \bar{\Gamma} := \dim \bar{\mathbf{S}}$.

We obtain for $\ell \gg 0$ that

$$(27) \quad \dim_{\ell} \bar{\Gamma}_{\ell}^{\text{geo}} = \dim_{\ell} \bar{\mathbf{S}}_{\ell}(\mathbb{F}_{\ell})^+ = \dim_{\ell} \bar{\mathbf{S}}_{\ell}(\mathbb{F}_{\ell}) = \dim \bar{\mathbf{S}}_{\ell} = \dim \mathbf{G}_{\ell}^{\text{geo}}$$

by Step I, [No87, 3.6(v)], [HL14, Prop. 3(iii)], and (26) respectively for each equality. Recall the universal covering $\pi : \mathbf{G}_{\ell}^{\text{sc}} \rightarrow \mathbf{G}_{\ell}^{\text{geo}}$. Since (a) the kernel and cokernel of $\pi^{-1}(\Gamma_{\ell}^{\text{geo}}) \rightarrow \Gamma_{\ell}^{\text{geo}}$ are abelian and (b) the kernel of $\Gamma_{\ell}^{\text{geo}} \twoheadrightarrow \bar{\Gamma}_{\ell}^{\text{geo}}$ is pro-solvable (via the reduction map $\text{GL}(H^i(Y_{\bar{x}_0}, \mathbb{Z}_{\ell})) \rightarrow \text{GL}(\bar{V}_{\ell})$ for $\ell \gg 0$), we obtain by the properties of \dim_{ℓ} that for $\ell \gg 0$,

$$(28) \quad \dim_{\ell} \pi^{-1}(\Gamma_{\ell}^{\text{geo}}) \stackrel{(a)}{=} \dim_{\ell} \Gamma_{\ell}^{\text{geo}} \stackrel{(b)}{=} \dim_{\ell} \bar{\Gamma}_{\ell}^{\text{geo}} \stackrel{(27)}{=} \dim \mathbf{G}_{\ell}^{\text{geo}} =: g.$$

Step V. Let Δ_{ℓ} be a maximal compact subgroup of $\mathbf{G}_{\ell}^{\text{sc}}(\mathbb{Q}_{\ell})$ that contains $\pi^{-1}(\Gamma_{\ell}^{\text{geo}})$. By [Ti79, 3.2], Δ_{ℓ} is the stabilizer $\mathbf{G}_{\ell}^{\text{sc}}(\mathbb{Q}_{\ell})^x$ of a vertex x in the *Bruhat-Tits building* of $\mathbf{G}_{\ell}^{\text{sc}}/\mathbb{Q}_{\ell}$. There exists a smooth affine group scheme \mathcal{G} over \mathbb{Z}_{ℓ} and an isomorphism ι from the generic fiber of \mathcal{G} to $\mathbf{G}_{\ell}^{\text{sc}}$ such that $\iota(\mathcal{G}(\mathbb{Z}_{\ell})) = \mathbf{G}_{\ell}^{\text{sc}}(\mathbb{Q}_{\ell})^x$ [Ti79, 3.4.1]. As $\mathbf{G}_{\ell}^{\text{sc}}$ is simply-connected semisimple, the special fiber $\mathcal{G}_{\mathbb{F}_{\ell}}$ is connected [Ti79, 3.5.2]. The maximal compact subgroup Δ_{ℓ} is *hyperspecial* if and only if $\mathcal{G}_{\mathbb{F}_{\ell}}$ is reductive [Ti79, 3.8.1], in which case it has the same root datum as the generic fiber [SGA3, XXII, 2.8]. Since (c) the kernel of the reduction map $r : \mathcal{G}(\mathbb{Z}_{\ell}) \rightarrow \mathcal{G}(\mathbb{F}_{\ell})$ is pro-solvable and (d) \mathcal{G} is smooth over \mathbb{Z}_{ℓ} , we obtain by the properties of \dim_{ℓ} that for $\ell \gg 0$,

$$(29) \quad \dim_{\ell} r(\pi^{-1}(\Gamma_{\ell}^{\text{geo}})) \stackrel{(c)}{=} \dim_{\ell} \pi^{-1}(\Gamma_{\ell}^{\text{geo}}) \stackrel{(28)}{=} g = \dim \mathbf{G}_{\ell}^{\text{sc}} \stackrel{(d)}{=} \dim \mathcal{G}_{\mathbb{F}_{\ell}}.$$

Since the special fiber $\mathcal{G}_{\mathbb{F}_{\ell}}$ is connected, $\mathcal{G}_{\mathbb{F}_{\ell}}$ is semisimple for $\ell \gg 0$ by (29) and [HL14, Thm. 4(iv)]. It follows from above that Δ_{ℓ} is hyperspecial and $\mathcal{G}_{\mathbb{F}_{\ell}}$ is simply-connected semisimple for $\ell \gg 0$. For any connected algebraic group $\bar{\mathbf{G}}$ of dimension g defined over \mathbb{F}_{ℓ} , the order of $\bar{\mathbf{G}}(\mathbb{F}_{\ell})$ satisfies

$$(30) \quad (\ell - 1)^g \leq |\bar{\mathbf{G}}(\mathbb{F}_{\ell})| \leq (\ell + 1)^g$$

by [No87, Lem 3.5]. Hence, there exists a constant $c(g) \geq 1$ depending only on g such that for $\ell \gg 0$,

$$(31) \quad \frac{(\ell - 1)^g}{c(g)} \leq |r(\pi^{-1}(\Gamma_{\ell}^{\text{geo}}))| \stackrel{\text{subgp}}{\leq} |\mathcal{G}(\mathbb{F}_{\ell})| \stackrel{(30)}{\leq} (\ell + 1)^g,$$

where the first inequality follows by considering (29), (30), the properties of \dim_{ℓ} in Step IV, and the orders of finite simple groups of Lie type in characteristic ℓ [St67, §9] (e.g., $|\text{PSL}_k(\ell)| = \frac{1}{(k, \ell-1)} \ell^{k(k-1)/2} (\ell^2 - 1)(\ell^3 - 1) \cdots (\ell^k - 1)$). Since $g := \dim \mathbf{G}_{\ell}^{\text{geo}} \leq n^2$ for all ℓ , the index $[\mathcal{G}(\mathbb{F}_{\ell}) : r(\pi^{-1}(\Gamma_{\ell}^{\text{geo}}))] \leq C(n)$ (a constant depending only on n) for $\ell \gg 0$. Since $\mathcal{G}_{\mathbb{F}_{\ell}}$ is simply-connected semisimple, $\mathcal{G}(\mathbb{F}_{\ell})$ is generated

by the subset of order ℓ elements $\mathcal{G}(\mathbb{F}_\ell)[\ell]$ when $\ell \gg 0$ (see the proof of [HL14, Thm. 4]). Since $\mathcal{G}(\mathbb{F}_\ell)[\ell]$ belongs to $r(\pi^{-1}(\Gamma_\ell^{\text{geo}}))$ for $\ell \gg C(n)$, the equality $r(\pi^{-1}(\Gamma_\ell^{\text{geo}})) = \mathcal{G}(\mathbb{F}_\ell)$ holds for $\ell \gg C(n)$. Therefore, the subgroup $\pi^{-1}(\Gamma_\ell^{\text{geo}}) \subset \mathcal{G}(\mathbb{Z}_\ell)$ surjects onto $\mathcal{G}(\mathbb{F}_\ell)$ under the reduction map r for $\ell \gg 0$. By the main theorem of [Va03], this implies $\pi^{-1}(\Gamma_\ell^{\text{geo}}) = \mathcal{G}(\mathbb{Z}_\ell) = \Delta_\ell$ for $\ell \gg 0$, which is hyperspecial maximal compact in $\mathbf{G}_\ell^{\text{geo}}(\mathbb{Q}_\ell)$. \square

Corollary 3.1. *For all sufficiently large ℓ , the identity component of the algebraic geometric monodromy group $\mathbf{G}_\ell^{\text{geo}}$ is unramified over \mathbb{Q}_ℓ .*

Proof. Since a connected reductive group $\mathbf{G}/\mathbb{Q}_\ell$ is unramified if and only if $\mathbf{G}(\mathbb{Q}_\ell)$ contains a hyperspecial maximal compact subgroup [Mi92, §1], $\mathbf{G}_\ell^{\text{sc}}$ is unramified for $\ell \gg 0$ by Theorem 3. Since $\pi : \mathbf{G}_\ell^{\text{sc}} \twoheadrightarrow (\mathbf{G}_\ell^{\text{geo}})^\circ$ is surjective, the identity component $(\mathbf{G}_\ell^{\text{geo}})^\circ$ is unramified for $\ell \gg 0$. \square

Remark 3.2. *Assuming Φ_ℓ is semisimple for all ℓ , then $\mathbf{G}_\ell^\circ \times \mathbb{C} \subset \text{GL}_{V_\ell \times \mathbb{C}}$ is independent of ℓ by [Ch04] and [Ka99] (see Step V of Theorem 2). Corollary 3.1 is a necessary condition for the existence of a common \mathbb{Q} -form of $\{\mathbf{G}_\ell^\circ \subset \text{GL}_{V_\ell}\}_\ell$.*

4. SEMISIMPLICITY

In this section, we give two examples of $\{\Phi_\ell\}$ such that the hypothesis of Theorem 3 holds. It suffices to show by the lemma below that the restriction of ϕ_ℓ to a normal subgroup of $\pi_1^{\text{et}}(X_{\bar{\mathbb{F}}_q})$ (i.e., by taking a connected Galois étale cover of X) is semisimple for $\ell \gg 0$.

Lemma 4.1. [HL15b, Lemma 3.6] *Let F be a field, G a finite group, H a normal subgroup of G such that $[G : H]$ is non-zero in F , and V a finite dimensional F -representation of G . Then V is semisimple if and only if its restriction to H is so.*

Example 1. Suppose the fibers of $f : Y \rightarrow X$ are curves or abelian varieties. Then the hypothesis of Theorem 3 holds.

Proof. Step I. When X is a curve, ϕ_ℓ is factored through by a Galois representation of $K(X)$, the function field of X . When the fibers of f are abelian varieties, the conclusion follows directly from the Tate conjecture of abelian varieties over function fields [Za74a, Za74b] (see also [FW84, Ch. VI§3], [LP95, Thm. 3.1(iii)]). When the fibers are curves, the conclusion follows from above and the fact that a smooth curve and its Jacobian variety have isomorphic H^1 representations.

Step II. For general X , we may first assume $\mathbf{G}_\ell^{\text{geo}}$ is connected for all ℓ by taking a connected Galois étale cover [LP95, Prop. 2.2(i)]. By [Ka99] and [Ch04] (see Step V of Theorem 2), there exists a smooth geometrically connected curve C of X such that the algebraic geometric monodromy group associated to $Y \times_X C \rightarrow C$ is also equal to $\mathbf{G}_\ell^{\text{geo}}$ for all ℓ . Denote the images of $\pi_1^{\text{ét}}(C_{\bar{\mathbb{F}}_q})$ and $\pi_1^{\text{ét}}(X_{\bar{\mathbb{F}}_q})$ in respectively $\text{GL}(V_\ell)$ and $\text{GL}(\bar{V}_\ell)$ by

$$\begin{aligned}\Lambda_\ell^{\text{geo}} &\subset \Gamma_\ell^{\text{geo}} \subset \text{GL}(V_\ell); \\ \bar{\Lambda}_\ell^{\text{geo}} &\subset \bar{\Gamma}_\ell^{\text{geo}} \subset \text{GL}(\bar{V}_\ell).\end{aligned}$$

We may assume $\bar{\Gamma}_\ell^{\text{geo}}$ is generated by its order ℓ elements for $\ell \gg 0$ by [CT15]. Since $\pi^{-1}(\Lambda_\ell^{\text{geo}}) \subset \pi^{-1}(\Gamma_\ell^{\text{geo}})$ are compact subgroups of $\mathbf{G}_\ell^{\text{sc}}(\mathbb{Q}_\ell)$ and $\pi^{-1}(\Lambda_\ell^{\text{geo}})$ is hyperspecial maximal compact in $\mathbf{G}_\ell^{\text{sc}}(\mathbb{Q}_\ell)$ for $\ell \gg 0$ by Step I and Theorem 3, we have $\pi^{-1}(\Lambda_\ell^{\text{geo}}) = \pi^{-1}(\Gamma_\ell^{\text{geo}})$ for $\ell \gg 0$. Hence, the index $[\Gamma_\ell^{\text{geo}} : \Lambda_\ell^{\text{geo}}]$ is bounded by some constant C (depending on $n = \dim V_\ell$) for $\ell \gg 0$. It follows that $[\bar{\Gamma}_\ell^{\text{geo}} : \bar{\Lambda}_\ell^{\text{geo}}] \leq C$ for $\ell \gg 0$ via the reduction map $\text{GL}(H^i(Y_{\bar{x}_0}, \mathbb{Z}_\ell)) \rightarrow \text{GL}(\bar{V}_\ell)$. This implies that the order ℓ elements of $\bar{\Gamma}_\ell^{\text{geo}}$ belong to $\bar{\Lambda}_\ell^{\text{geo}}$ when $\ell \gg C$. Since $\bar{\Gamma}_\ell^{\text{geo}}$ is generated by its order ℓ elements and $\bar{\Lambda}_\ell$ is semisimple on \bar{V}_ℓ for $\ell \gg 0$, $\bar{\Gamma}_\ell^{\text{geo}}$ is semisimple on \bar{V}_ℓ for $\ell \gg 0$. \square

Example 2. Identify Γ_ℓ^{geo} as a subgroup of $\text{GL}(H^i(Y_{\bar{x}_0}, \mathbb{Z}_\ell)) = \text{GL}_n(\mathbb{Z}_\ell)$ for $\ell \gg 0$. Suppose there exists a connected semisimple subgroup $\mathbf{G} \subset \text{GL}_{n, \mathbb{Q}}$ such that $(\mathbf{G}_\ell^{\text{geo}})^\circ = \mathbf{G} \times \mathbb{Q}_\ell$ in $\text{GL}_{n, \mathbb{Q}_\ell}$ and

$$\Gamma_\ell^{\text{geo}} \cap (\mathbf{G}_\ell^{\text{geo}})^\circ \subset \mathbf{G}(\mathbb{Z}_\ell) \subset \text{GL}_n(\mathbb{Z}_\ell)$$

for $\ell \gg 0$. Then the hypothesis of Theorem 3 holds.

Proof. Step I. By taking a connected Galois étale cover, we may assume $\mathbf{G}_\ell^{\text{geo}}$ is connected for all ℓ [LP95, Prop. 2.2(i)] and $\bar{\Gamma}_\ell^{\text{geo}} = (\bar{\Gamma}_\ell^{\text{geo}})^+$ for $\ell \gg 0$ [CT15]. The closed subgroup $\mathbf{G} \subset \text{GL}_{n, \mathbb{Q}}$ can be extended to a closed subgroup scheme $\mathbf{G}_{\mathbb{Z}[\frac{1}{N}]} \subset \text{GL}_{n, \mathbb{Z}[\frac{1}{N}]}$ smooth over $\mathbb{Z}[\frac{1}{N}]$ for some sufficiently divisible integer N . Let $\mathbf{G}_{\mathbb{F}_\ell} \subset \text{GL}_{n, \mathbb{F}_\ell}$ be the base change to \mathbb{F}_ℓ for $\ell \gg 0$. Since $\bar{\Gamma}_\ell^{\text{geo}} \subset \mathbf{G}_{\mathbb{F}_\ell}$ for $\ell \gg 0$, we obtain

$$\dim_{\bar{\mathbb{Q}}_\ell}(V_\ell^{\otimes m} \otimes \bar{\mathbb{Q}}_\ell)^{\mathbf{G}} \stackrel{\text{Lem. 2.1}}{\leq} \dim_{\bar{\mathbb{F}}_\ell}(\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\mathbf{G}_{\mathbb{F}_\ell}} \leq \dim_{\bar{\mathbb{F}}_\ell}(\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\bar{\Gamma}_\ell^{\text{geo}}}$$

for $\ell \gg 0$. Since $\dim_{\bar{\mathbb{Q}}_\ell}(V_\ell^{\otimes m})^{\pi_1^{\text{ét}}(X_{\bar{\mathbb{F}}_q})} = \dim_{\bar{\mathbb{Q}}_\ell}(V_\ell^{\otimes m} \otimes \bar{\mathbb{Q}}_\ell)^{\mathbf{G}}$ as Γ_ℓ^{geo} is Zariski dense in \mathbf{G} , we obtain

$$(33) \quad \dim_{\bar{\mathbb{F}}_\ell}(\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\mathbf{G}_{\mathbb{F}_\ell}} = \dim_{\bar{\mathbb{F}}_\ell}(\bar{V}_\ell^{\otimes m} \otimes \bar{\mathbb{F}}_\ell)^{\bar{\Gamma}_\ell^{\text{geo}}}$$

for $\ell \gg 0$ by (32) and Theorem 2. Since $\mathbf{G}_{\mathbb{F}_\ell}$ is connected semisimple for $\ell \gg 0$, the natural representation $i_\ell : \mathbf{G}_{\mathbb{F}_\ell} \rightarrow \text{GL}(\bar{V}_\ell \otimes \bar{\mathbb{F}}_\ell)$ is semisimple

for $\ell \gg 0$ [La95b]. Hence, it suffices to prove that for all $\ell \gg 0$, the restriction of any irreducible $\bar{\mathbb{F}}_\ell$ -subrepresentation $W_{\bar{\mathbb{F}}_\ell}$ of i_ℓ to $\bar{\Gamma}_\ell^{\text{geo}}$ is still irreducible as $\bar{\Gamma}_\ell^{\text{geo}} \subset \mathbf{G}_{\bar{\mathbb{F}}_\ell}$.

Step II. Suppose $W_{\bar{\mathbb{F}}_\ell}$ is a direct summand of i_ℓ . Then for any $m \in \mathbb{N}$, we have

$$(34) \quad \dim_{\bar{\mathbb{F}}_\ell}(W_{\bar{\mathbb{F}}_\ell}^{\otimes m})^{\mathbf{G}_{\bar{\mathbb{F}}_\ell}} = \dim_{\bar{\mathbb{F}}_\ell}(W_{\bar{\mathbb{F}}_\ell}^{\otimes m})^{\bar{\Gamma}_\ell^{\text{geo}}}$$

when ℓ is sufficiently large by (33). Suppose $\bar{\Gamma}_\ell^{\text{geo}}$ is not irreducible on $W_{\bar{\mathbb{F}}_\ell}$. Then there exists a k -dimensional subrepresentation $U_{\bar{\mathbb{F}}_\ell}$ of $\bar{\Gamma}_\ell^{\text{geo}}$ and $k < \dim W_{\bar{\mathbb{F}}_\ell} \leq n$ holds. By (34) and (the proof of) Corollary 2.3,

$$(35) \quad \dim_{\bar{\mathbb{F}}_\ell}(\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\mathbf{G}_{\bar{\mathbb{F}}_\ell}} = \dim_{\bar{\mathbb{F}}_\ell}(\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\bar{\Gamma}_\ell^{\text{geo}}}$$

holds when $\ell \gg 0$. Since $\bar{\Gamma}_\ell^{\text{geo}} \subset \mathbf{G}_{\bar{\mathbb{F}}_\ell}$ for $\ell \gg 0$,

$$(36) \quad (\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\mathbf{G}_{\bar{\mathbb{F}}_\ell}} = (\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\bar{\Gamma}_\ell^{\text{geo}}}$$

holds when $\ell \gg 0$. Since $\bar{\Gamma}_\ell^{\text{geo}}$ is generated by its order ℓ elements for $\ell \gg 0$, $\text{Alt}^k U_{\bar{\mathbb{F}}_\ell}$ is one-dimensional and belongs to $(\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\bar{\Gamma}_\ell^{\text{geo}}}$ when $\ell \gg 0$ by construction. Thus, $\text{Alt}^k U_{\bar{\mathbb{F}}_\ell} \subset (\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\mathbf{G}_{\bar{\mathbb{F}}_\ell}}$ by (36) which is impossible. Indeed, let $\{v_1, \dots, v_k\}$ be a basis of $U_{\bar{\mathbb{F}}_\ell}$ and $Z_{\bar{\mathbb{F}}_\ell} \neq 0$ a complement of $U_{\bar{\mathbb{F}}_\ell}$ in $W_{\bar{\mathbb{F}}_\ell}$. Since $\mathbf{G}_{\bar{\mathbb{F}}_\ell}$ is irreducible on $W_{\bar{\mathbb{F}}_\ell}$, there exists $x \in \mathbf{G}_{\bar{\mathbb{F}}_\ell}(\bar{\mathbb{F}}_\ell)$ that does not preserve $U_{\bar{\mathbb{F}}_\ell}$. Then we have the following equations

$$\begin{aligned} x \cdot v_1 &= u_1 + z_1 \\ x \cdot v_2 &= u_2 + z_2 \\ &\vdots \\ x \cdot v_k &= u_k + z_k, \end{aligned}$$

where the notation is defined so that $u_i \in U_{\bar{\mathbb{F}}_\ell}$ and $z_i \in Z_{\bar{\mathbb{F}}_\ell}$ for $1 \leq i \leq k$. We may assume $\{z_1, \dots, z_h\}$ is a non-empty maximal linearly independent subset of $\{z_1, \dots, z_k\}$. If $\text{Alt}^k U_{\bar{\mathbb{F}}_\ell} \subset (\text{Alt}^k W_{\bar{\mathbb{F}}_\ell})^{\mathbf{G}_{\bar{\mathbb{F}}_\ell}}$, then

$$(37) \quad x \cdot (v_1 \wedge \dots \wedge v_k) = (u_1 + z_1) \wedge \dots \wedge (u_k + z_k) \in \text{Alt}^k U_{\bar{\mathbb{F}}_\ell}.$$

Since $z_1 \neq 0$, we obtain $k > 1$ by (37). Since we have the decomposition

$$(38) \quad \text{Alt}^k(U_{\bar{\mathbb{F}}_\ell} \oplus Z_{\bar{\mathbb{F}}_\ell}) = \bigoplus_{i+j=k} \text{Alt}^i U_{\bar{\mathbb{F}}_\ell} \otimes \text{Alt}^j Z_{\bar{\mathbb{F}}_\ell},$$

we have $z_1 \wedge \dots \wedge z_k = 0$ by (37), which is the same as $h < k$. We may assume z_{h+1}, \dots, z_k are all equal to zero by the fact that $\{z_1, \dots, z_h\}$ is a maximal linearly independent subset of $\{z_1, \dots, z_k\}$ and replacing $\{v_1, \dots, v_h, v_{h+1}, \dots, v_k\}$ with a suitable basis $\{v_1, \dots, v_h, v'_{h+1}, \dots, v'_k\}$. It follows that $\{z_1, \dots, z_h, u_{h+1}, \dots, u_k\}$ is linearly independent because x is

invertible. Therefore, $u_{h+1} \wedge \cdots \wedge u_k \wedge z_1 \wedge \cdots \wedge z_h$ is non-zero, which is absurd by $z_{h+1} = \cdots = z_k = 0$, (37), and (38). This implies that $\bar{\Gamma}_\ell^{\text{geo}}$ cannot have a k -dimensional subrepresentation of $W_{\bar{\mathbb{F}}_\ell}$ when ℓ is sufficiently large. Since there are finitely many k less than $\dim W_{\bar{\mathbb{F}}_\ell} \leq n$, we conclude that $\bar{\Gamma}_\ell^{\text{geo}}$ is irreducible on $W_{\bar{\mathbb{F}}_\ell}$ and thus semisimple on $\bar{V}_\ell \otimes \bar{\mathbb{F}}_\ell$ if ℓ is sufficiently large. \square

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